Dissipation and Phase Slip in Confined Superfluid Helium: Evidence for an Equilibrium Distribution of Vortices

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Abstract

In this talk we discuss the effects of the phase slip due to the thermal creation of vortices in confined liquid Helium-four below the lambda point. In a narrow tube there is a finite probability of thermal activation of a vortex in the superfluid Helium, with the vortex not parallel to the axis of the tube. In the presence of a heat flow along the tube, the vortices experience the Magnus forse, which prevents exact cancellation of the motion of thermally activated vortices traversing the cross-section of the tube in opposite directions. Each crossing of the tube by a vortex causes a 2π phase slip of the superfluid order parameter along the tube. A temperature gradient results, which is proportional to the rate of phase slip, thus yielding a non-vanishing thermal resistivity below the bulk lambda point. The calculated temperature dependence compares well with experimental data, thereby providing indirect evidence of the presence of vortices in thermal equilibrium.

Key words: confined superfluid, helium, phase slip, thermal resistivity, vortex.

1 INTRODUCTION

Many years ago, in his celebrated studies of the theory of superfluid Helium-four, Feynman suggested that the lambda transition might result from the presence of vortex loops and rings. These would become more abundant with increasing temperature, leading to the developement of a kind of entangled network that could destroy the quantum mechanical order. This picture of vortices being the primary agent, or even sole agent, for the lambda transition has continued, through the years, to receive a great deal of attention. On the other hand, the standard approach to the critical dynamics for a system with a two-dimensional (i. e., quantum mechanical) order parameter has been based on fluctuations within the framework of the Ginzburg-Landau free energy functional. These are generally in the form of plane-wave-like Fourier components. The thrust of this paper is to attempt to reconcile the two different schools of thought by demonstrating that, although there can be no question of the success of the Ginzburg-Landau approach, there is clear experimental evidence of the presence of an equilibrium ensemble of vortices in the superfluid state close to the lambda point. The evidence comes from a phenomenon that is described briefly in a recent publication [1] and which is reviewed in Section II below. When the superfluid is contained inside a tube of small diameter a slight, strongly temperature dependent residual resistance is observed just below the bulk lambda transition point. This we have been able to attribute to the motion of vortices across the cross section of the tube. In the present paper we want to turn the situation around and regard our treatment as sufficiently reliable that we can use it as a "diagnostic tool" which indicates to us that there is a population of thermally excited vortices. The theory of this phenomenon procedes in two steps: 1)To first approximation, a barrier to the motion of the vortices is placed at the saddle-point configuration so as to lead to a "quasi-equilibrium" distribution which is distorted by the transverse forces that are exerted on the vortices as a consequence of the heat flow. This perturbed distribution is fixed near the inside wall of the tube, where usual equilibrium prevails for the vortices of small length. Consequently, the probability distribution on one side of the barrier is larger than its normal equilibrium value while on the other side it is smaller; 2)This asymmetry then serves as a kind of boundary condition near the barrier when the latter is lowered to permit some vortex motion across it, as described by Kramers theory for this kind of process. The computation leads to a rate of passage over the saddle-point barrier that is proportional to the equilibrium probability of finding a thermally excited vortex in the saddle-point configuration. For simplicity, the effect on this probability from fluctuations in the shape of the vortices, studied by us to some extent [2], will not be dealt with here. One important detail worth emphasizing, nevertheless: the probability has the form of a Boltzmann factor, the exponent of which contains a logarithmic dependence on the temperature. This is a robust and inescapable feature of the theory which invites experimental test, doubtless, however, requiring the sensitivity and precision to be achieved only in a microgravity environment.

2 DISSIPATION AND PHASE SLIP IN CONFINED SUPERFLUID HELIUM

2.1 When is a superfluid not superfluid?

Consider the axial flow of heat in a sample of liquid ^{4}He that is contained in a long right circular cylinder of radius r_0, as studied experimentally by Kahn and Ahlers [3] and reviewed recently by Ahlers [4]. Depending upon the value of r_0, this lateral confinement has a pronounced effect on the thermal conductivity which, in the unconfined bulk fluid, would diverge at the lambda point. This divergence is well understood [5] to be a consequence of the diverging mean lifetime of the longest wavelength fluctuation modes of the order parameter. Such a critical slowing down is characteristic of a second order phase transition, and is related in turn to the divergence at the critical point of ξ, the correlation length. The confinement that is under consideration here serves to interrupt this familiar critical behavior of ξ once it has grown to be comparable to r_0. The effect on the thermal conductivity in the vicinity of the bulk lambda point, as well as below it to some extent, has been studied by Hausmann [6]. The work that we report here applies to temperatures further below, where ξ shrinks to values much smaller than r_0. Although the ^{4}He is then in a state of broken symmetry, has developed a well-defined order parameter, and has become almost perfectly superfluid, the thermal resistance does not, in fact, drop to zero. This follows from the normal-fluid viscosity μ_n and from the "sticking" boundary condition at the inner surface of the confining cylinder that is imposed on

v_n, the axial component of the velocity of the normal fluid. The Poiseuille velocity profile for v_n, corresponding to the vanishing of v_n at the boundary, implies a pressure gradient proportional to the mean value of v_n, and, thus, proportional to the heat current. In steady state there is no acceleration of the superfluid and the fountain-effect relationship applies, yielding

$$gradT = \frac{1}{n\sigma} gradP \ , \tag{2.1}$$

where n and σ are the atomic density and the entropy per ^{4}He atom, respectively. Because these parameters, as well as μ_n, do not exhibit any pronounced critical variation, Eq. (2.1) contributes a non-critical background resistivity, as mentioned by Ahlers [4] and also discussed elsewhere [7], [8].

2.2 Vortices

The non-critical background thermal resistivity expressed by Eq. (2.1) depends upon the above assertion of a well-defined order parameter, and tacitly ignores any fluctuations in the order parameter. For $\xi \ll r_0$, these are obviously quite localized and can be expected to appear and promptly disappear, causing no long-term effect on the steady-state behavior of the order parameter. Thermally excited vortices, on the other hand, can have a quite different effect. These topological singularities have an intrinsic stability in their lateral structure (i.e., transverse to the vortex core) and can grow or disappear only by changes in their length. To the extent that they can be as long as $2r_0$, thus spanning across a diameter of the tube in a saddle-point configuration [8], they can migrate across the tube and with each such event produce a phase change in the order parameter of 2π. In thermal equilibrium, we can anticipate that such migrations will occur with equal probability in opposing directions so that no net superfluid acceleration is required for maintaining steady state flow, thus leaving Eq. (2.1) as the sole origin of the temperature gradient. The cancellation is, however, no longer complete when the system is perturbed by the normal fluid flowing past the vortices. The computation of this non-equilibrium effect that we will be carrying out in the following subsections depends in an important and essential way on the equilibrium probability of a saddle-point vortex. We denote this by $\rho_{eq}(0)$ and proceed now to examine it in detail.

The superfluid velocity encircling a quantized vortex has the space dependence

$$v_s \propto r^{-1} \ , \tag{2.2}$$

where r is the distance from the center of the vortex core. The kinetic energy density is consequently of the form

$$\frac{\rho_s}{2} v_s^2 \propto \frac{\rho_s}{2} \cdot \frac{1}{r^2} \tag{2.3}$$

ρ_s being the superfluid density. An integration with upper and lower cut-offs at the cylinder radius and at the correlation length, respectively, yields, for a vortex length equal to the full diameter,

$$G_0 \propto 2r_0 \int \frac{\rho_s}{2} \frac{d^2 r}{r^2} \propto r_0 \rho_s \ln \frac{r_0}{\xi} \ . \tag{2.4}$$

(A more careful attention to these cutoffs will be provided elsewhere [7].) We need the Clow-Reppy critical temperature dependence,

$$\rho_s / \rho = 2.4 |t|^{\nu} = 2.4 \xi_0 / \xi \tag{2.5}$$

in terms of ξ, with $\xi_0 = 0.7\ \mathring{A}$, and we take further numerical factors from the detailed work of Rayfield and Reif [9]. We consequently obtain for the Gibbs free energy of a saddle-point vortex divided by the temperature (in units of energy)

$$\Omega = \frac{G_0}{T} = 0.6\frac{r_0}{\xi}\ln\frac{r_0}{\xi} = 0.6X\ln X \ , \tag{2.6}$$

where

$$X \equiv r_0/\xi \ . \tag{2.7}$$

The intrinsic probability is, therefore, given by the Boltzmann factor

$$\rho_{eq}(0) = \exp\left(-\Omega\right) \tag{2.8}$$

2.3 Quasi-Equilibrium

The heat current density is proportional to the local axial normal fluid velocity, according to

$$Q = T\rho\sigma_n v_n \cong T_\lambda \rho_\lambda \sigma v_n \ , \tag{2.9}$$

where, close to the lambda point, the temperature, entropy, and mass density can be approximated by their lambda-point values. Having only heat flow and no mass flow requires a counterflow of the superfluid at velocity v_s according to

$$\rho_s v_s = \rho_n \bar{v}_n \ , \tag{2.10}$$

with the bar indicating the cross-sectional average. ¿From Eq. (2.5) this becomes

$$v_s = \frac{\rho_n}{\rho_s}\bar{v}_n \cong \frac{\rho}{\rho_s}\bar{v}_n = \frac{\xi}{2.4\xi_0}\bar{v}_n \ . \tag{2.11}$$

In terms of the distribution of the ^{4}He atoms in momentum space this displaces the delta function (representing the discrete contribution of the order parameter) away from the origin by

$$\frac{mv_s}{\hbar} = \frac{m\xi}{\hbar\xi_0} v_n \propto \xi Q \ , \tag{2.12}$$

where m and $2\pi\hbar$ are the ^{4}He mass and Planck's constant, respectively. As the lambda point is approached from below for fixed Q, Eq. (2.12) signifies that the displacement will increase until it becomes comparable to ξ^{-1}, the breadth of the fluctuation cloud surrounding the delta function. This will lead to a departure from non-linearity. The limiting criterion for linearity thus becomes the scaling relation

$$Q \leq Q_{non-lin} \propto \xi^{-2} \propto t^{2\nu} \tag{2.13}$$

for a given temperature. For a given heat current, Q, this translates into

$$|t| \geq |t|_{min} \propto Q^{\frac{1}{2\nu}} \ . \tag{2.14}$$

Equation (2.14) also applies for $t > 0$, with a somewhat similar theoretical basis [10, 11]. In the present context, the counterflow can be expected to generate additional vortices [12] that are not taken into account by our theory. The actual experimental upper bound on Q for linear heat flow may, therefore, be smaller than $Q_{non-lin}$ of Eq. (2.13).

An essential assumption in our treatment of the movement of the vortices is that the inner wall of the confining cylinder is sufficiently smooth on the mesoscopic scale of $\xi \gg \xi_0$ that there is no pinning of the vortices to the walls. We, thus arrive at the picture of the vortices being carried along with the superfluid, with the normal fluid passing by them at relative velocity $v_s + v_n$ or simply v_s (because $v_n \ll v_s$). Let $l(x)$ be the half-length of a "minimal" vortex, i.e. one in the shape of an arc of a circle, lying in a plane perpendicular to the axis of the cylinder, and with both ends normal to the cylinder wall. The coordinate x is the distance from the arc center to the cylinder axis, so that $x = 0$ specifies the saddle point configuration, with $l(0) = r_0$. In the absence of heat flow we would have the equilibrium probability

$$\sigma_{eq}(x) = \exp\left[-l(x)\Omega/r_0\right] , \tag{2.15}$$

reducing to Eq. (2.8) for $x = 0$.

For the purposes of this calculation, we can treat F, the Magnus force per unit length of the normal fluid acting on a vortex, as a conservative force described by the potential energy

$$V_M(x) = -FA(x) , \tag{2.16}$$

with $A(x)$ being $(\pi/2)r_0^2$ minus the area enclosed by the arc. In other words, $A(x)$ is the area lying between the arc and a diameter: thus, $A(x)$ vanishes for $x = 0$ and is equal to $2r_0x$ for $|x| \ll r_0$. It is evident that our choice of sign for x is such that the Magnus force of Eq. (2.16) will tend to cause the vortices to move in the direction of increasing x. It is convenient, however, to imagine a model in which the vortices are not free to move. A quasi-equilibrium would then be developed, of modified probability

$$\rho_{q-eq}(x) = \exp\left[-l(x)\Omega/r_0 + A(x)F/T\right] \cong \left[1 + A(x)\frac{F}{T}\right]\rho_{eq}(x)$$

$$= \rho_{eq}(x) + \Delta\rho_{eq}(x) \ . \tag{2.17}$$

The perturbation caused by the Magnus force is, thus, to first order in F,

$$\Delta\rho_{eq}(x) = A(x)\frac{F}{T}\rho_{eq}(x) \ . \tag{2.18}$$

Of special interest will be the asymmetry introduced for the very short vortices close to the cylinder wall, of $l \ll r_0$ and $|x| \cong r_0$ for which

$$\Delta\rho_{eq}(\pm r_0) = \pm\frac{\pi r_0^2}{2T}F\rho_{eq}(\pm r_0) \tag{2.19}$$

and

$$\Delta\rho_{eq}(r_0) - \Delta\rho_{eq}(-r_0) = \frac{\pi r_0^2}{T}F\rho_{eq}(\pm r_0) \tag{2.20}$$

(because $\rho_{eq}(r_0) = \rho_{eq}(-r_0)$, by definition).

2.4 Saddle-point passage

Returning to Eq. (2.17) and substituting : $l(x) = (r_0^2 - x^2)^{1/2} = r_0 - x^2/(2r_0) + \ldots$, we see that near the saddle point

$$\rho_{q-eq}(x) \cong \rho_{q-eq}(0)\exp\left[x^2/x_0^2\right] \tag{2.21}$$

with a characteristic length defined by

$$x_0 = \sqrt{2}r_0\Omega^{-1/2} \tag{2.22}$$

Aided by the work on phase slip in superconductors of McCumber and Halperin [13] and of Ambegaokar and Halperin [14] as well as by that of Langer [15] on nucleation, we apply the method of Kramers [16] by taking $\rho_{q-eq}(x)$ as a jumping-off point. Allowing the vortices now to drift, we obtain the probability distribution for the non-equilibrium system as

$$\rho_{non-eq}(x) = \rho_{q-eq}(x) + \Delta\rho_{q-eq}(x) \ , \tag{2.23}$$

where the additional perturbation from the transport properties has an asymmetric form

$$\Delta\rho_{q-eq}(x) = f(x)\rho_{q-eq}(x) \; . \tag{2.24}$$

An exact solution for $f(x)$ from the Kramers transport equation by the method of "variation of constants" is readily obtained [7] in the form of the error function. It suffices here, however, to adopt a slightly less quantitative and more heuristic approach by limiting ourselves to the asymptotic behavior

$$f(x) \mid_{|x| \gg x_0} = f_{asymp}(x) = -f_0 sgn(x) \; . \tag{2.25}$$

This sets in as soon as the vortex is further from the saddle point than the characteristic distance x_0. This parameter is a measure of the sharpness of the saddle point and determines the steepness of the descent down into the valleys on either side. The connection of the probability current for the steady state drift, J, to the constant f_0 is determined by the diffusion coefficient, D, and is most easily evaluated at the minimum of $\rho_{q-eq}(x)$. Thus, from Eqs. (2.23) and (2.24),

$$J = -D\frac{\delta\rho_{non-eq}(x)}{\delta x} = -D\frac{\delta\Delta\rho_{q-eq}(x)}{\delta x} = -D\rho_{q-eq}(x)\frac{df}{dx} \; . \tag{2.26}$$

Because f changes to its full asymptotic value in an interval of the size x_0, we have

$$J = D\frac{f_0}{x_0}\rho_{q-eq}(0) \; , \tag{2.27}$$

up to a numerical factor of $O(1)$.

It remains to fix f_0 by means of an appropriate boundary condition. The final solution to the transport problem is the perturbed probability distribution function, $\rho_{non-eq}(x)$. Returning to Eqs. (2.23-2.24), and substituting (2.17) yields

$$\begin{aligned}\rho_{non-eq}(x) &= [1+f(x)]\,\rho_{q-eq}(x) \cong [1+f(x)]\left[1+A(x)\frac{F}{T}\right]\rho_{eq}(x) \\ &\cong \left[1+f(x)+A(x)\frac{F}{T}\right]\rho_{eq}(x)\;, \end{aligned} \tag{2.28}$$

to first order. For the required boundary condition we adopt

$$\rho_{non-eq}(\pm r_0) = \rho_{eq}(\pm r_0)\;, \tag{2.29}$$

which expresses the idea that the Magnus force perturbs the distribution inside the cylinder, but not at its walls. We assume that at the walls the usual relaxation processes are strong enough to maintain unperturbed equilibrium. This boundary condition requires that the two first-order terms in Eq. (2.28) cancel, yielding

$$f(\pm r_0) = -\frac{F}{T}A(\pm r_0) = \mp\frac{\pi}{2}\frac{Fr_0^2}{T} \tag{2.30}$$

which, by substitution from Eq. (2.25), becomes

$$f_0 = \frac{\pi}{2}\frac{Fr_0^2}{T}\;. \tag{2.31}$$

In substituting Eq. (2.31) into Eq. (2.27), it is usual to recall the Einstein fluctuation-dissipation theorem : $\mu = D/T$, which expresses μ, the mobility of a vortex, in terms of its diffusion coefficient. By substitution from Eq.

(2.22) and by the introduction of $F_{tot} = 2r_0 F$ for the Magnus force acting on a saddle-point vortex, Eq. (2.27) becomes

$$J = \Omega^{1/2} \mu F_{tot} \rho_{eq}(0) \ , \tag{2.32}$$

again with the neglect of numerical factors of $O(1)$. Aside from the temperature dependent dimensionless factor $\Omega^{1/2}$, this result has the simple physical interpretation of vortices being dragged across the saddle-point at rate μF_{tot}.

3 SUMMARY

In this paper we have presented a theory which provides a "diagnostic tool" which indicates the existence of a population of thermally excited vortices in confined liquid Helium-four below the lambda point. Serving as a fingerprint of the presence of this population of vortices is our predicted temperature dependence of the thermal resistivity in the tube filled with liquid ^{4}He . This dependence is expected to have the form of a Boltzmann factor, the exponent of which logarithmically depends on the temperature. A direct experimental test of this theoretical prediction should be possible, although it would seem to require the sensitivity and precision to be achieved only in a microgravity environment.

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